HIGHER INDEX FANOS with FINITELY MANY BIRATIONAL AUTOS

jt. with Nathan Chen

X a proj. variety $/k=\bar{k}$.

$\mathrm{Bir}\, X = \left\{ \begin{matrix} \text{bir. autos.} \\ \text{of } X \end{matrix} \right\}$.

Q1 What aspects of geometry of X control $\mathrm{Bir}\, X$?

Q2 When is $\mathrm{Bir}\, X$ finite?

Non-example.

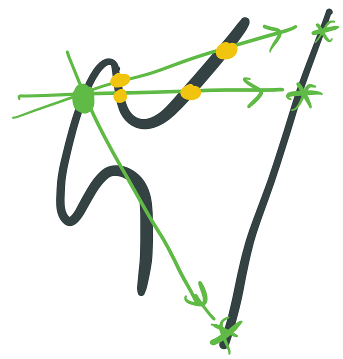
$X = \mathbb{P}^n$: $\mathrm{Bir}\, X$ is large!
$(n > 1)$

Thm: (Noether-Castelnuovo)
$\mathrm{Bir}(\mathbb{P}^2)$ is generated by PGL_3 and $\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$
$[x:y:z] \longmapsto [\frac{1}{x} : \frac{1}{y} : \frac{1}{z}]$.

Thm: (Lin-Shinder)
$\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ is not generated by regularizable elements when $n \geq 4$.

Ex. X a cubic hypersurface.

Then $\mathrm{Bir}\,X$ is large.



Proj. from any point.

$\Downarrow$

2:1 cover of $\mathbb{P}^n$ $\Rightarrow$ covering involution.

ALTERNATIVE Project from a line. $\ell \subseteq X$.

Given a conic bdle str.

X and conics have
$\vdots$ automorphisms
$\mathbb{P}^{n-1}$

ALTERNATIVE Project from an arbitrary line. $\ell \not\subseteq X$.

$X \dashrightarrow \mathbb{P}^{n-1}$.
ell. fibration + section [k 3 points]

(ELLIPTIC + gives many others).

On the other hand...

If X general type, then:

$$\phi : X \dashrightarrow X.$$

gives rise to:

$$\phi^* : H^0(mK_X) \longrightarrow H^0(mK_X).$$

So ϕ is equiv. to an auto:

$$\begin{tikzcd}
X^{can} \arrow[r, "\phi^{can}"'] & X^{can} \\
X \arrow[u, dashed] \arrow[r, dashed, "\phi"'] & X \arrow[u, dashed]
\end{tikzcd}$$

$\rightsquigarrow$ $\mathrm{Bir}\,X$ is finite.

[Similarly: If X satisfies $K_X \equiv 0$ AND $\rho = 1$
then $\phi : X \dashrightarrow X$ is actually regular.

(1) Show it's defined in codim 1.

(2) analyze contracted curves in closure of graph. [Ref: Lin-Shinder])

Q If X is Fano, when is $\mathrm{Bir}\,X$ finite?

RESTRICTING to HYPERSURFACES.

If $X \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$ has degree $n+1$ $(n \geq 3)$
then X is birationally super-rigid.

↳ $X \to *$ is the only Mori fiber space structure.

$\Rightarrow$ Any birational auto. is an isomorphism $\Rightarrow \mathrm{Bir}\,X$ finite.

(+ much more)

(DUE to Fano-Segre-Iskovskikh-Manin
-Pukhlikov-Corti-Cheltsov-DeFernex
-Ein-Mustaţă-Zhang)

Uses "Noether-Fano method"

Pukhlikov: if $n \geq 14$ and $d = n$ then
$\mathrm{Bir}\,X = \mathrm{Aut}\,X$ is finite.

Recall, the index of a Fano variety is the max r s.t.

$$-K_X \equiv rH$$

Cartier divisor.

Q. Are there examples of large index Fano varieties s.t. Bir X is finite?

Main THM (Chen–S.)

If char $k = p > 0$: there are Fano varieties (w/ terminal sings) of arbitrarily large index with Bir $X = \{1\}$.

Kollár's n-1 forms.

In breakthrough paper "Nonrational Hypersurfaces" Kollár observes/constructs Fano varieties many $(n-1)$-forms.

μ_p-covers: $X \xrightarrow[p:1]{\mu} Y.$

Recall Given $s \in \Gamma(Y, \mathcal{L}^{\otimes p})$ $\exists$ a μ_p-cover of Y "branched" at $(s=0)$.

$$\begin{tikzcd}
 & t \arrow[r] & t^{\otimes p} & \\
X = \mu^{-1}Y \subseteq & \mathbb{L} \arrow[r, "\mu"'] \arrow[dr, "\pi"'] & \mathbb{L}^{\otimes p} \arrow[d] & \supseteq Y. \\
 & & Y \arrow[ur, bend right, "s"'] &
\end{tikzcd}$$

Constructing the terms:

X has local eqn:

$$t^p = s(y_1, \dots, y_n) \subseteq \mathbb{L}$$

$$\begin{tikzcd}
 & \mathcal{O}_X(-X) = (\pi^* \mathcal{L}^{-p})|_X \arrow[d] \arrow[dl, bend right] & \\
(\pi^* \Omega_Y)|_X \arrow[r] \arrow[d] & \Omega_{\mathbb{L}}|_X \arrow[r] \arrow[d] & \Omega_{\mathbb{L}/Y}|_Y = (\pi^* \mathcal{L})|_X. \\
Q \arrow[r] & \Omega_X &
\end{tikzcd}$$

NOTICE: $\det Q \subset \Lambda^{n-1}\Omega_X$ can be ample + gg'd even if X is Fano.

$\exists$ a resolution:

$$Z \longrightarrow X.$$

Theorem (Chen-S)

Let $k=\bar{k}$ alg. closed of char $=p>0$.
$n\geq 3$ (if $p=2$, assume n even).
Let $Y\subseteq \mathbb{P}^{n+1}_k$ be a smooth degree e hypersurface. Fix $d>0$ and assume

$$(p-1)d\leq n-e\leq pd-3.$$

If X is a μ_p-cover over a general section of $\mathcal{O}_Y(pd)$, then:

$$H^0(Z,\Lambda^{n-1}\Omega_Z)\cong H^0(X,\det Q)$$

$$\cong H^0(Y,\omega_Y(pd))$$

$$\overset{(1)}{\left(\mathcal{O}_Y(m)\quad m>0\right)}$$

Claim. $\det Q$ is birationally equivalent.

Defn. Let $\phi : X \dashrightarrow X$. There is a pullback:

$$\phi^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X)$$

(notice: map is defined away from codim ≥ 2).

Say: $\mathcal{L} \in \mathrm{Pic}\, X$ admits a $\mathrm{Bir}\, X$-equivariant structure, if $\forall\, \phi \in \mathrm{Bir}\, X$

$\exists$ a choice:

$$\phi^*\mathcal{L} \xrightarrow[\cong]{\lambda_\phi} \mathcal{L}.$$

s.t. $\forall\, \phi_1, \phi_2$:

$$\begin{tikzcd}
(\phi_1 \circ \phi_2)^* \mathcal{L} = \phi_1^*(\phi_2^* \mathcal{L}) \arrow[r, "\phi_2^*(\lambda_{\phi_1})"] \arrow[dr, "\lambda_{\phi_1 \circ \phi_2}"'] & \phi_2^* \mathcal{L} \arrow[d, "\lambda_{\phi_2}"] \\
& \mathcal{L}
\end{tikzcd}$$

compatibility

BASIC PROPERTIES

(1) Bir X-equivariant line bundles form a group.

(2) ∃ a natural hom:

$$\mathrm{Pic}_{\mathrm{Bir}(X)}[X] \longrightarrow \mathrm{Pic}\, X.$$

(3) If $\mathcal{L}$ is Bir X-equivariant, and $H^0(X,\mathcal{L}) \neq 0$, then ∃ a repn:

$$\rho : \mathrm{Bir}\, X \longrightarrow GL(H^0(X,\mathcal{L})^\vee)$$

s.t. $\forall \phi \in \mathrm{Bir}\, X$:

$$\begin{tikzcd}
X \arrow[r, dashed, "\phi"] \arrow[d, dashed, "i"] & X \arrow[d, dashed, "i"] \\
\mathbb{P}(H^0(X,\mathcal{L}^\vee)) \arrow[r, "\rho(\phi)"'] & \mathbb{P}(H^0(X,\mathcal{L}^\vee))
\end{tikzcd}$$

commutes.

(4) X smooth, proj. If the image of

$$H^0(X,\Lambda^i\Omega_X)\otimes\mathcal{O}_X \longrightarrow \Lambda^i\Omega_X$$

is a line bundle $\mathcal{L}$ then $\mathcal{L}$ is birationally equivariant.

REMAINING

In the construction have:

$$X \xrightarrow[\mu]{} Y.$$

Have: $\omega_Y(pd)$ very ample.

$+\ \mu^*\omega_Y(pd)$ $\mathrm{Bir}\,X$-equivariant.

$\Rightarrow$ any $\phi \in \mathrm{Bir}\,X$ descends to an automorphism of Y.

In fact $\mathrm{Bir}\,X \longrightarrow \mathrm{Aut}\,Y$ is injective (a bijection on k-points!)

Easy to find Y with $\mathrm{Aut}\,Y = \{1\}$.